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Problem 3123.Proposed by Joe Hovard, Portales,NM,USA.

Let a,b,c be the sides of a triangle. Show that

$$\frac{abc(a+b+c)^2}{a^2+b^2+c^2} \geq 2abc + \prod_{cyclic}(a+b-c).$$

Solution by Arkady Alt , San Jose ,California, USA.

Let F,s,R,r denote the area, semiperimeter, circumradius and inradius, respectively, of triangle. Using corellations $abc = 4FR,\ ab+bc+ca = s^2+r^2+4Rr$ and $F = rs$ we can to give to the original inequality "more" geometrical form:

$$\frac{4FR \cdot 4s^2}{a^2+b^2+c^2} \geq 8FR + \frac{8F^2}{s} \Leftrightarrow \frac{2Rs^2}{a^2+b^2+c^2} \geq R+r \Leftrightarrow \frac{2R}{R+r} \geq \frac{a^2+b^2+c^2}{s^2} = \frac{4s^2-2(s^2+r^2+4Rr)}{s^2} \Leftrightarrow \frac{R}{R+r} \leq 1-\frac{r^2+4Rr}{s^2} \Leftrightarrow \frac{r^2+4Rr}{s^2} \leq \frac{r}{R+r} \Leftrightarrow$$

(**1**) $s^2 \leq (R+r)(4R+r)$.

So, original inequality equivalent to (1), which immediatelly follows from Gerretsen's inequality $s^2 \leq 4R^2+4Rr+3r^2$ and Eyler's inequality $2r \leq R$.

Really, $(R+r)(4R+r)-(4R^2+4Rr+3r^2) = Rr-2r^2 = r(R-2r) \geq 0$.

Remark. Another solution in CRUX vol.33.n.2

Problem 3125.Proposed by Walther Janous, Ursulinengymnasium,Insbruck, Austria.(Solutions in CRUX vol.33.n.3 and nobody solved c*))

Let m_a, h_a and w_a denote the lengths of the median, the altitude, and the internal angle bisector, respectively to side a in $\triangle ABC$.

(**a**) Show that

$$\sum_{cyclic}\frac{b^2+c^2}{m_a} \leq 12R.$$

(**b**) $\displaystyle\sum_{cyclic}\frac{b^2+c^2}{h_a} \geq 12R.$

(**c**) ★ Determine the range of

$$\frac{1}{R}\sum_{cyclic}\frac{b^2+c^2}{w_a}.$$

Solution by Arkady Alt , San Jose ,California, USA.

(**a**) Let R and d_a be distance, respectively, circumradius and distance from the circumcenter to side a. Then by triangle inequality $m_a \leq R+d_a$ and, since $d_a = \sqrt{R^2-\frac{a^2}{4}}$ then we obtain:

$$m_a - R \leq \sqrt{R^2-\frac{a^2}{4}} \Leftrightarrow m_a^2-2m_aR+R^2 \leq R^2-\frac{a^2}{4} \Leftrightarrow$$

$$4m_a^2-8m_aR+a^2 \leq 0 \Leftrightarrow 2(b^2+c^2)-a^2-8m_aR+a^2 \leq 0 \Leftrightarrow b^2+c^2 \leq 4m_aR \Leftrightarrow$$

(**1**) $\dfrac{b^2+c^2}{m_a} \leq 4R$.

Hence, $\displaystyle\sum_{cyclic}\frac{b^2+c^2}{m_a} \leq \sum_{cyclic}4R = 12R.$

(**b**) Let F be area of triangle $\triangle ABC$. Since $4FR = abc$ and $\displaystyle\sum_{cyclic}a(b^2+c^2) =$

$(a+b+c)(ab+bc+ca)-3abc$, then $\sum_{cyclic} \frac{b^2+c^2}{h_a} \geq 12R \Leftrightarrow$

$\sum_{cyclic} \frac{a(b^2+c^2)}{2F} \geq 12R \Leftrightarrow \sum_{cyclic} a(b^2+c^2) \geq 6abc \Leftrightarrow$

$(a+b+c)(ab+bc+ca) \geq 9abc$,

where latter inequality follows from $a+b+c \geq 3\sqrt[3]{abc}$ and

$ab+bc+ca \geq 3\sqrt[3]{a^2b^2c^2}$.

Remark. Another solutions to a and b in CRUX vol.33.n.3 and c* remains usolved.

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